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A cardinal generalization of C^*-embedding and its applications

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Abstract

As for extending real-valued continuous functions or continuous pseudometrics on a subspace to the whole space, notions of z-, C^*-, C-, P- and P^γ-embeddings are known. As a cardinal generalization of z-embedding, Blair defined in 1985 the notion of z_γ-embedding with $\gamma \geqslant \omega$, where z_ω-embedding coincides with z-embedding. On the other hand, since P^ω-embedding equals C-embedding, P^γ-embedding can be also regarded as a cardinal generalization of C-embedding. Recently Ohta asked if a cardinal generalization of C^*-embedding can be defined so that this property plus U^ω-embedding is equal to P^γ-embedding, and it is itself equals C^*-embedding in case $\gamma = \omega$. In this paper, we give a cardinal generalization of C^*-embedding, called $(P^*)^\gamma$-embedding, and answer this problem. As a characterization of $(P^*)^\gamma$-embedding, we show that $(P^*)^\gamma$-embedding naturally admits its description by using continuous maps from a subspace into the hedgehog with γ spines. We also give a new extension-like property called weak z_γ-embedding, with which z-(respectively C^*-, C- or U^ω-) embedding equals z_γ- (respectively $(P^*)^\gamma$-, P^γ- or U^γ-) embedding.

Keywords: C^*-embedding; P^γ-embedding; z_γ-embedding; C-embedding; z-embedding; Well-embedding; U^γ-embedding; Hedgehog; Approximation

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1. Introduction

Throughout this paper by a space we mean a topological space, and γ denotes an infinite cardinal number.

We state the definitions of some familiar extension properties of a subspace of a space; basic references are [1,5,6,9,16]. Let X be a space and A a subspace. Then A is C-*embedded* (respectively C^*-*embedded*) in X if every real-valued (respectively bounded real-valued) continuous function on A can be extended to a real-valued continuous

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function over X. A is *P-embedded* (respectively *P^γ-embedded*) in X if every continuous pseudometric (γ-separable continuous pseudometric) on A can be continuously extended over X [18]. A is *z-embedded* in X if every zero-set in A is the intersection of A with some zero-set in X. In case $\gamma = \omega$, P^ω-embedding coincides with C-embedding [1, Theorem 16.4]. In [3] Blair defined the notion of z_γ-embedding (see Section 2 for the definition) which also coincides with z-embedding in case $\gamma = \omega$. Thus, P^γ-embedding or z_γ-embedding can be viewed as a cardinal generalization of C-embedding or z-embedding, respectively. However, as for C^*-embedding, no such generalization has been known so far.

In discussing extension properties and several other topological properties, Ohta recently asked in [16, Problem 2.5] if an extension property can be defined so that this property plus U^ω-embedding (see Section 3 for the definition of U^γ-embedding) is equal to P^γ-embedding, and it is itself equal C^*-embedding in case $\gamma = \omega$; he suggested to the author that this extension property might be called $(P^*)^\gamma$-embedding.

In the present paper, we define an extension property, which we call $(P^*)^\gamma$-embedding, and show that this property satisfies the requirements in Ohta's problem above and further satisfies "$(P^*)^\gamma$ is equal to $C^* + z_\gamma$". Recall Blair's theorem "P^γ is equal to $C + z_\gamma$" [3, p. 21]. In Section 3, $(P^*)^\gamma$-embedding is characterized by continuous maps into the hedgehog with γ spines. In [17] and [3], P^γ-embedding and z_γ-embedding have been characterized in terms of these maps (see Lemmas 2.1 and 2.2).

In Section 4, approximations of continuous hedgehog-valued maps on a subspace to continuous hedgehog-valued ones over the whole space will be used to describe P^γ-, $(P^*)^\gamma$- and z_γ-embedding. In particular, first, we show that the differences among these embeddings depend only on the number of spines of $J(\gamma)$ on which approximations can be made (Theorems 4.1 and 4.2). Secondly, again using approximations, we give other characterizations of P^γ-embedding and z_γ-embedding (Theorems 4.7 and 4.9); for z_γ-embedding, the result was stated in Tsukada [19], but the proof contains several indistinct parts.

In Section 5, extending z_γ-embedding, we define another type of extension-like property called weak z_γ-embedding. It is shown that under weak z_γ-embedding z- (respectively C^*-, C- or U^ω-) embedding equals z_γ- (respectively $(P^*)^\gamma$-, P^γ- or U^γ-) embedding. We note that no such property has been known under which all of these equalities hold.

2. Preliminaries

Let us first recall the definition of the hedgehog with γ spines (e.g., [5]). Let $I_\xi = [0,1] \times \{\xi\}$ for every $\xi < \gamma$. Define the equivalence relation E on $\bigcup_{\xi<\gamma} I_\xi$ as $(x, \xi_1) E (y, \xi_2)$ whenever $x = y = 0$ or ($x = y$ and $\xi_1 = \xi_2$). Denote by $J(\gamma)$ the set of all equivalence classes of E and define a metric on $J(\gamma)$ as follows:

$$\rho\big((x,\xi_1),(y,\xi_2)\big) = \begin{cases} |x-y| & \text{if } \xi_1 = \xi_2, \\ x+y & \text{if } \xi_1 \neq \xi_2 \end{cases}$$

for every (x,ξ_1), $(y,\xi_2)\in J(\gamma)$. We call $J(\gamma)$ with this metric the hedgehog with γ spines. θ stands for the class of $J(\gamma)$ consisting of $(0,\xi)$, $\xi<\gamma$. And we define the continuous function $\Pi: J(\gamma)\to[0,1]$ by $\Pi((x,\xi))=x$ and the map $j:(J(\gamma)-\{\theta\})\to\gamma$ by $j((x,\xi))=\xi$. In particular, we admit in this paper the case that the number of spines is finite and denote by $J(n)$ the hedgehog with n spines.

Let X be a space and A a subspace. For a cover $\mathcal{U}$ of A and a collection $\mathcal{V}$ of subsets of X which covers A, $\mathcal{V}\wedge A<\mathcal{U}$ denotes "$\{V\cap A \mid V\in\mathcal{V}\}$ refines $\mathcal{U}$". The characterizations of P^γ-embedding below are due to Alò and Shapiro [1], Engelking [5], Morita [12] and Przymusiński [17].

Proposition 2.1. *Let X be a space and A a subspace. The following statements are equivalent*:

(1) *A is P^γ-embedded in X;*
(2) *for every locally finite cover $\mathcal{U}$ of cozero-sets of A with $|\mathcal{U}|\leqslant\gamma$ there exists a locally finite* (*or equivalently, σ-locally finite*) *cover $\mathcal{V}$ of cozero-sets of X such that $\mathcal{V}\wedge A<\mathcal{U}$;*
(3) *for every continuous map $f:A\to J(\gamma)$ there exists a continuous map $g:X\to J(\gamma)$ such that $g|A=f$;*
(4) *for every continuous map $f:A\to Y$ into a Čech complete* AR *space* (= AR *for metric spaces*) *Y with weight $Y\leqslant\gamma$ there exists a continuous map $g:X\to Y$ such that $g|A=f$.*

Next we mention z_γ-embeddings. Let X be a space and A a subspace. By Blair [3], A is said to be *z_γ-embedded* in X if every normal open cover $\mathcal{U}$ of A with $|\mathcal{U}|\leqslant\gamma$, there exists a cozero-set G of X containing A and a normal open cover $\mathcal{V}$ of G such that $\mathcal{V}\wedge A<\mathcal{U}$. If A is z_γ-embedded in X for every γ, A is said to be *z_∞-embedded* in X. By [3, Theorem 3.5], z_ω-embedding coincides with z-embedding. A is said to be *well-embedded* in X if A is completely separated from any zero-set disjoint from A. By [3, Theorem 4.6], A is z_γ-embedded and well-embedded in X if and only if A is P^γ-embedded in X; we briefly write this fact by "$P^\gamma=z_\gamma+$ well". By this fact, we have other facts "$C=z+$ well" [4, Corollary 3.6.B] and "$C=C^*+$ well" [6, p. 19].

For collections $\{F_\alpha \mid \alpha\in\Omega\}$ and $\{G_\alpha \mid \alpha\in\Omega\}$ of subsets of a space, $\{G_\alpha \mid \alpha\in\Omega\}$ is said to be an *expansion* of $\{F_\alpha \mid \alpha\in\Omega\}$ if $F_\alpha\subset G_\alpha$ for every $\alpha\in\Omega$. And a collection $\mathcal{E}$ of subsets of X is said to be *uniformly locally finite* (respectively *uniformly discrete*) in X if there exists a normal open cover $\mathcal{U}$ of X such that each member of $\mathcal{U}$ intersects at most finitely many (respectively at most one) members of $\mathcal{E}$. By Morita [13] or Ohta [15] (respectively Blair [3]), $\{E_\alpha \mid \alpha\in\Omega\}$ is uniformly locally finite (respectively uniformly discrete) in X if and only if there exists a locally finite (respectively discrete) collection $\{G_\alpha \mid \alpha\in\Omega\}$ of cozero-sets of X and a collection $\{Z_\alpha \mid \alpha\in\Omega\}$ of zero-sets of X such that $E_\alpha\subset Z_\alpha\subset G_\alpha$ for every $\alpha\in\Omega$. From now on, we often use the following two facts: The first is that, by Morita and Hoshina [14], if a collection $\{F_\alpha \mid \alpha\in\Omega\}$ of zero-sets has a locally finite expansion of cozero-sets (i.e., $\{F_\alpha \mid \alpha\in\Omega\}$ is uniformly locally finite collection of zero-sets), then $\bigcup\{F_\alpha \mid \alpha\in\Omega\}$ is a zero-set. The second is that, by Blair [3], if $\{H_\xi \mid \xi < \gamma\}$ is a disjoint collection of open subsets of X and $\bigcup_{\xi<\gamma} H_\xi = h^{-1}((0,1])$ for some continuous function $h : X \to [0,1]$, then the map $g : X \to J(\gamma)$ is continuous, where g is defined by $g(x) = (h(x), \xi)$ if $x \in H_\xi$ $(\xi < \gamma)$; $g(x) = \theta$ otherwise. Other terminology are referred to [1,5,6,9].

The following fact due to Hoshina [8] is frequently used: A is z_γ-embedded in X if and only if for every locally finite cover $\mathcal{U}$ of cozero-sets of A with $|\mathcal{U}| \leqslant \gamma$ there exists a σ-locally finite collection $\mathcal{V}$ of cozero-sets of X such that $\mathcal{V}$ covers A and $\mathcal{V} \wedge A < \mathcal{U}$. We now give below some other equivalent conditions of z_γ-embedding. In particular, (1) $\Leftrightarrow$ (4) of Lemma 2.2 was proved by Blair [3, Theorem 3.8]. Here we give a proof for the sake of convenience.

Lemma 2.2. *Let X be a space and A a subspace. Then the following statements are equivalent*:

(1) *A is z_γ-embedded in X*;

(2) *for every discrete collection $\{G_\alpha \mid \alpha < \gamma\}$ of cozero-sets of A and every collection $\{F_\alpha \mid \alpha < \gamma\}$ of zero-sets of A with $F_\alpha \subset G_\alpha$ for each $\alpha < \gamma$, for each $i < \omega$ there exists a locally finite collection $\mathcal{H}_i = \{H_{i\alpha} \mid \alpha < \gamma\}$ of cozero-sets of X such that $H_{i\alpha} \cap A \subset G_\alpha$ for each $\alpha < \gamma$, $i < \omega$ and $F_\alpha \subset \bigcup\{H_{i\alpha} \mid i < \omega\}$ for each $\alpha < \gamma$*;

(3) *for every disjoint collection $\{G_\alpha \mid \alpha < \gamma\}$ of cozero-sets of A for which $\bigcup\{G_\alpha \mid \alpha < \gamma\}$ is a cozero-set of A, there exists a disjoint collection $\{H_\alpha \mid \alpha < \gamma\}$ of cozero-sets of X such that $H_\alpha \cap A = G_\alpha$ for each $\alpha < \gamma$ and $\bigcup\{H_\alpha \mid \alpha < \gamma\}$ is a cozero-set of X*;

(4) *for every continuous map $f : A \to J(\gamma)$, there exists a continuous map $g : X \to J(\gamma)$ such that $g^{-1}((0,1]) \times \{\xi\}) \cap A = f^{-1}((0,1] \times \{\xi\})$ for each $\xi < \gamma$.*

Proof. To prove (1) $\Rightarrow$ (2), assume A is z_γ-embedded in X. Let $\{G_\alpha \mid \alpha < \gamma\}$ be a discrete collection of cozero-sets of A and $\{F_\alpha \mid \alpha < \gamma\}$ a collection of zero-sets of A with $F_\alpha \subset G_\alpha$ for each $\alpha < \gamma$. Then $\{G_\alpha \mid \alpha < \gamma\} \cup \{A - \bigcup_{\alpha<\gamma} F_\alpha\}$ is a locally finite cover of cozero-sets of A. By the fact mentioned above by Hoshina, there exists a σ-locally finite collection $\mathcal{V}$ of cozero-sets of X such that $\mathcal{V}$ covers A and $\mathcal{V} \wedge A$ refines $\{G_\alpha \mid \alpha < \gamma\} \cup \{A - \bigcup_{\alpha<\gamma} F_\alpha\}$. We can express $\mathcal{V} = \bigcup_{i<\omega} \mathcal{V}_i$, where $\mathcal{V}_i$ is locally finite in X for each $i < \omega$. Put $H_{i\alpha} = \bigcup\{V \in \mathcal{V}_i \mid V \cap F_\alpha \neq \emptyset\}$ for each $\alpha < \gamma$ and $i < \omega$. Then each $H_{i\alpha}$ is a cozero-set of X. Since $\{G_\alpha \mid \alpha < \gamma\}$ is disjoint, $\{H_{i\alpha} \mid \alpha < \gamma\}$ is locally finite for each $i < \omega$. And we can verify that $H_{i\alpha} \cap A \subset G_\alpha$ for each $\alpha < \gamma$, $i < \omega$ and $F_\alpha \subset \bigcup\{H_{i\alpha} \mid i < \omega\}$ for each $\alpha < \gamma$. Hence (2) is satisfied.

To prove (2) $\Rightarrow$ (1), assume (2) is satisfied. Let $\mathcal{U}$ be a locally finite cover of cozero-sets of A with $|\mathcal{U}| \leqslant \gamma$. Take a cover $\{G_\alpha^n \mid \alpha < \gamma, n < \omega\}$ of cozero-sets of A which refines $\mathcal{U}$ and a cover $\{F_\alpha^n \mid \alpha < \gamma,\ n < \omega\}$ of zero-sets of A such that $\{G_\alpha^n \mid \alpha < \gamma\}$ is discrete in A for each $n < \omega$ and $F_\alpha^n \subset G_\alpha^n$ for each $\alpha < \gamma,\ n < \omega$. From the assumption, there exists a locally finite collection $\mathcal{H}_i^n = \{H_{i\alpha}^n \mid \alpha < \gamma\}$ of cozero-sets of X for each $i < \omega$ satisfying that $H_{i\alpha}^n \cap A \subset G_\alpha^n$ for each $\alpha < \gamma,\ i < \omega$ and $F_\alpha^n \subset \bigcup\{H_{i\alpha}^n \mid i < \omega\}$ for each $\alpha < \gamma$. Hence $\{H_{i\alpha}^n \mid \alpha < \gamma;\ i, n < \omega\}$ is a σ-locally finite collection of cozero-sets of X which covers A and satisfies $\{H_{i\alpha}^n \mid \alpha < \gamma;\ i, n < \omega\} \wedge A < \mathcal{U}$. Hence by the result of Hoshina again, A is z_γ-embedded in X.

To prove (2) $\Rightarrow$ (3), assume (2) to be satisfied. Let $\{G_\alpha \mid \alpha < \gamma\}$ be a disjoint collection of cozero-sets of A and $\bigcup\{G_\alpha \mid \alpha < \gamma\}$ is a cozero-set of A. Let $f : A \to [0,1]$ be a continuous function with $\bigcup\{G_\alpha \mid \alpha < \gamma\} = f^{-1}((0,1])$. Now, fix an $n \in \mathbb{N}$. Then, $\{G_\alpha \cap f^{-1}((1/(n+1),1]) \mid \alpha < \gamma\}$ is a discrete collection of cozero-sets of A. We shall show $G_\alpha \cap f^{-1}([1/n,1])$ is a zero-set of A for each $\alpha < \gamma$. Since $\bigcup\{G_\beta \mid \beta \neq \alpha;\ \beta < \gamma\}$ is a cozero-set of $\bigcup_{\beta<\gamma} G_\beta$ and $\bigcup_{\beta<\gamma} G_\beta$ is a cozero-set of A, we have $\bigcup\{G_\beta \mid \beta \neq \alpha;\ \beta < \gamma\}$ is a cozero-set of A. Since

$$G_\alpha \cap f^{-1}\left(\left[\frac{1}{n},\ 1\right]\right) = \left(A - \bigcup\{G_\beta \mid \beta \neq \alpha;\ \beta < \gamma\}\right) \cap f^{-1}\left(\left[\frac{1}{n},\ 1\right]\right),$$

$G_\alpha \cap f^{-1}([1/n,1])$ is a zero-set of A.

Therefore for $\{G_\alpha \cap f^{-1}((1/(n+1),1]) \mid \alpha < \gamma\}$ and $\{G_\alpha \cap f^{-1}([1/n,1]) \mid \alpha < \gamma\}$, from (2), there exists a locally finite collection $\mathcal{H}_i^n = \{H_{i\alpha}^n \mid \alpha < \gamma\}$ of cozero-sets of X for each $i < \omega$ such that $H_{i\alpha}^n \cap A \subset G_\alpha \cap f^{-1}((1/(n+1),1])$ for each $\alpha < \gamma$, $i < \omega$ and $G_\alpha \cap f^{-1}([1/n,1]) \subset \bigcup\{H_{i\alpha}^n \mid i < \omega\}$ for each $\alpha < \gamma$. For each $\alpha < \gamma$ and $i < \omega$, let $g_{i\alpha}^n : X \to [0,1]$ be a continuous function such that $H_{i\alpha}^n = (g_{i\alpha}^n)^{-1}((0,1])$. Put $Z_{ji\alpha}^n = (g_{i\alpha}^n)^{-1}([1/j,\ 1])$ and $W_{ji\alpha}^n = (g_{i\alpha}^n)^{-1}((1/j,\ 1])$ for each $j \in \mathbb{N}$. Then $\{W_{ji\alpha}^n \mid \alpha < \gamma\}$ and $\{Z_{ji\alpha}^n \mid \alpha < \gamma\}$ are locally finite in X for each $j,n \in \mathbb{N}$ and $i < \omega$. Notice that $\{Z_{ji\alpha}^n \mid \alpha < \gamma\}$ has a locally finite expansion $\{W_{(j+1)i\alpha}^n \mid \alpha < \gamma\}$ of cozero-sets of X.

Since the indices $\{(n,i,j) \mid n,j \in \mathbb{N},\ i < \omega\}$ can be reordered into ω, there exist locally finite collections $\mathcal{Z}_k = \{Z_{k\alpha} \mid \alpha < \gamma\}$ $(k < \omega)$ of zero-sets of X and $\mathcal{W}_k = \{W_{k\alpha} \mid \alpha < \gamma\}$ $(k < \omega)$ of cozero-sets of X such that

$$\bigcup\{Z_{k\alpha} \mid k < \omega\} = \bigcup\{W_{k\alpha} \mid k < \omega\},$$

$$G_\alpha = \left(\bigcup\{W_{k\alpha} \mid k < \omega\}\right) \cap A$$

for each $\alpha < \gamma$ and $W_{k\alpha} \subset Z_{k\alpha}$ for each $\alpha < \gamma$, $k < \omega$. Let

$$H_\alpha = \bigcup_{k<\omega} \left(W_{k\alpha} - \bigcup\{Z_{j\beta} \mid \beta \neq \alpha,\ \beta < \gamma;\ j \leqslant k\}\right)$$

for each $\alpha < \gamma$. Then, $\{H_\alpha \mid \alpha < \gamma\}$ is a disjoint collection of cozero-sets of X with $H_\alpha \cap A = G_\alpha$ for each $\alpha < \gamma$. Since

$$\begin{aligned}\bigcup\{H_\alpha \mid \alpha < \gamma\} &= \bigcup_{\alpha<\gamma}\bigcup_{k<\omega} \left(W_{k\alpha} - \bigcup\{Z_{j\beta} \mid \beta \neq \alpha,\ \beta < \gamma;\ j \leqslant k\}\right) \\ &= \bigcup_{k<\omega}\bigcup_{\alpha<\gamma} \left(W_{k\alpha} - \bigcup\{Z_{j\beta} \mid \beta \neq \alpha,\ \beta < \gamma;\ j \leqslant k\}\right)\end{aligned}$$

and $\{W_{k\alpha} - \bigcup\{Z_{j\beta} \mid \beta \neq \alpha,\ \beta < \gamma;\ j \leqslant k\} \mid \alpha < \gamma\}$ is a locally finite collection of cozero-sets of X, $\bigcup\{H_\alpha \mid \alpha < \gamma\}$ is a cozero-set of X. Hence (3) is satisfied.

To prove (3) $\Rightarrow$ (4), assume (3) is satisfied. Let $f : A \to J(\gamma)$ be a continuous map. Then $\{f^{-1}((0,1] \times \{\xi\}) \mid \xi < \gamma\}$ is a disjoint collection of cozero-sets of A and $\bigcup\{f^{-1}((0,1] \times \{\xi\}) \mid \xi < \gamma\}$ is a cozero-set of A. From the assumption, there exists a disjoint collection $\{H_\xi \mid \xi < \gamma\}$ of cozero-sets of X such that $H_\xi \cap A = f^{-1}((0,1] \times \{\xi\})$ for each $\xi < \gamma$ and $\bigcup\{H_\xi \mid \xi < \gamma\}$ is a cozero-set of X. By the fact mentioned above, there exists a continuous map $g : X \to J(\gamma)$ such that $g^{-1}((0,1] \times \{\xi\}) \cap A = f^{-1}((0,1] \times \{\xi\})$ for each $\xi < \gamma$. Hence (4) is satisfied.

To prove (4) $\Rightarrow$ (2), assume (4) to be satisfied. Let $\{G_\alpha \mid \alpha < \gamma\}$ be a discrete collection of cozero-sets of A and $\{F_\alpha \mid \alpha < \gamma\}$ a collection of zero-sets of A with $F_\alpha \subset G_\alpha$ for each $\alpha < \gamma$. Since $\bigcup_{\alpha<\gamma} G_\alpha$ is a cozero-set of A and $\bigcup_{\alpha<\gamma} F_\alpha$ is a zero-set of A, there exists a continuous function $h : A \to [0,1]$ such that $\bigcup_{\alpha<\gamma} F_\alpha = h^{-1}(\{1\})$ and $\bigcup_{\alpha<\gamma} G_\alpha = h^{-1}((0,1])$. Define a continuous map $f : A \to J(\gamma)$ by

$$f(a) = \begin{cases} (h(a), \alpha) & \text{if } a \in G_\alpha \ (\alpha < \gamma), \\ \theta & \text{otherwise.} \end{cases}$$

And we have that

$$f^{-1}\big(\{\theta\}\big) = A - \bigcup_{\alpha<\gamma} G_\alpha,\ f^{-1}\big((0,1] \times \{\alpha\}\big) = G_\alpha \quad \text{and}$$
$$f^{-1}\big(\{1\} \times \{\alpha\}\big) = F_\alpha$$

for each $\alpha < \gamma$. From the assumption, there exists a continuous map $g : X \to J(\gamma)$ such that $g^{-1}((0,1] \times \{\alpha\}) \cap A = f^{-1}((0,1] \times \{\alpha\})$ for each $\alpha < \gamma$. Put $H_{n\alpha} = g^{-1}((1/n, 1] \times \{\alpha\})$ for each $\alpha < \gamma,\ n \in \mathbb{N}$. Then, we have that $\{H_{n\alpha} \mid \alpha < \gamma\}$ is a discrete collection of cozero-sets of X for each $n \in \mathbb{N}$, that $H_{n\alpha} \cap A \subset G_\alpha$ for each $\alpha < \gamma,\ n \in \mathbb{N}$ and that $F_\alpha \subset \bigcup\{H_{n\alpha} \mid n \in \mathbb{N}\}$ for each $\alpha < \gamma$. Thus (2) is satisfied. The proof is complete. □

Remark 2.3. Each of the following conditions is also equivalent to each of the statements of Lemma 2.2:

(2)′ *for every disjoint collection $\{G_\alpha \mid \alpha < \gamma\}$ of cozero-sets of A satisfying $\bigcup\{G_\alpha \mid \alpha < \gamma\}$ is a cozero-set of A and every collection $\{F_\alpha \mid \alpha < \gamma\}$ of zero-sets of A with $F_\alpha \subset G_\alpha$ for each $\alpha < \gamma$, the conclusion of* (2) *of Lemma* 2.2 *follows*;

(3)′ *for every discrete collection $\{G_\alpha \mid \alpha < \gamma\}$ of cozero-sets of A, the conclusion of* (3) *of Lemma* 2.2 *follows*.

3. $(P^*)^\gamma$-embedding

In this section, we give the definition of $(P^*)^\gamma$-embedding and some fundamental facts. Let X be a space and A a subspace. A is said to be $(P^*)^\gamma$*-embedded* in X if for every disjoint collection $\{U_\alpha \mid \alpha < \gamma\}$ of cozero-sets of A for which $\bigcup\{U_\alpha \mid \alpha < \gamma\}$ is a cozero-set of A and for any finitely many $\alpha_1, \ldots, \alpha_n$'s $< \gamma$ with any continuous functions $f_{\alpha_i} : A \to [0,1]$ $(i = 1, \ldots, n)$ satisfying $U_{\alpha_i} = f_{\alpha_i}^{-1}((0,1])$, there exists a disjoint collection $\{U_\alpha^* \mid \alpha < \gamma\}$ of cozero-sets of X and continuous functions $f_{\alpha_i}^* : X \to [0,1]$ $(i = 1, \ldots, n)$ such that the following conditions are satisfied;

(a) $\bigcup\{U_\alpha^* \mid \alpha < \gamma\}$ is a cozero-set of X,

(b) $U_\alpha^* \cap A = U_\alpha$ for every $\alpha < \gamma$, and

(c) $(f_{\alpha_i}^*)^{-1}((0,1]) = U_{\alpha_i}^*$ and $f_{\alpha_i}^* | A = f_{\alpha_i}$ for each $i = 1, \ldots, n$.

A subspace A of a space X is said to be P^**-embedded* in X if A is $(P^*)^\gamma$-embedded in X for every γ. Clearly, $(P^*)^\gamma$-embedding implies $(P^*)^{\gamma'}$-embedding if $\gamma \geqslant \gamma'$.

From (3) of Lemma 2.2 and the definition of $(P^*)^\gamma$-embedding, we have the following proposition.

Proposition 3.1. *If A is a $(P^*)^\gamma$-embedded subspace of a space X, then A is z_γ-embedded in X.*

The following theorem is our main result in this paper.

Theorem 3.2. *Let X be a space and A a subspace. Then the following statements are equivalent*:

(1) *A is $(P^*)^\gamma$-embedded in X;*

(2) *for every disjoint collection $\{U_\alpha \mid \alpha < \gamma\}$ of cozero-sets of A for which $\bigcup\{U_\alpha \mid \alpha < \gamma\}$ is a cozero-set of A and for any $\alpha_1 < \gamma$ with any continuous function $f_{\alpha_1} : A \to [0,1]$ satisfying $U_{\alpha_1} = f_{\alpha_1}^{-1}((0,1])$, there exists a disjoint collection $\{U_\alpha^* \mid \alpha < \gamma\}$ of cozero-sets of X and a continuous function $f_{\alpha_1}^* : X \to [0,1]$ such that the conditions* (a), (b) *and* (c) *of the definition of $(P^*)^\gamma$-embedding are satisfied;*

(3) *for every discrete collection $\{U_\alpha \mid \alpha < \gamma\}$ of cozero-sets of A, any finitely many $\alpha_1, \dots, \alpha_n$'s $< \gamma$ and any zero-set Z_{α_i} $(i = 1, \dots, n)$ of A satisfying $Z_{\alpha_i} \subset U_{\alpha_i}$ $(i = 1, \dots, n)$, there exists a disjoint collection $\{U_\alpha^* \mid \alpha < \gamma\}$ of cozero-sets of X and zero-sets $Z_{\alpha_i}^*$ $(i = 1, \dots, n)$ of X such that the conditions*
   (a) *$\bigcup\{U_\alpha^* \mid \alpha < \gamma\}$ is a cozero-set of X,*
   (b) *$U_\alpha^* \cap A = U_\alpha$ for every $\alpha < \gamma$, and*
   (c)′ *$Z_{\alpha_i}^* \cap A = Z_{\alpha_i}$ and $Z_{\alpha_i}^* \subset U_{\alpha_i}^*$ for each $i = 1, \dots, n$ are satisfied;*

(4) *for every discrete collection $\{U_\alpha \mid \alpha < \gamma\}$ of cozero-sets of A, any $\alpha_1 < \gamma$ and a zero-set Z_{α_1} of A satisfying $Z_{\alpha_1} \subset U_{\alpha_1}$, there exists a disjoint collection $\{U_\alpha^* \mid \alpha < \gamma\}$ of cozero-sets of X and a zero-set $Z_{\alpha_1}^*$ of X such that the conditions* (a), (b) *and* (c)′ *of* (3) *are satisfied;*

(5) *A is C^*- and z_γ-embedded in X;*

(6) *for every continuous map $f : A \to J(\gamma)$ and any finitely many $\xi_1, \dots, \xi_n$'s $< \gamma$, there exists a continuous map $g : X \to J(\gamma)$ such that the conditions*
   (i) *$g(a) = f(a)$ for each $a \in f^{-1}(\bigcup_{i=1}^n ((0,1] \times \{\xi_i\}))$, and*
   (ii) *$g^{-1}((0,1] \times \{\xi\}) \cap A = f^{-1}((0,1] \times \{\xi\})$ for each $\xi < \gamma$ are satisfied;*

(7) *for every continuous map $f : A \to J(\gamma)$ and any $\xi_1 < \gamma$, there exists a continuous map $g : X \to J(\gamma)$ such that* (i) *and* (ii) *of* (6) *are satisfied.*

Proof. (1) ⇒ (2), (1) ⇒ (3), (2) ⇒ (4) and (3) ⇒ (4) are easy to prove.

To prove (4) ⇒ (5), we assume (4) to be satisfied. To prove (2) of Lemma 2.2, let $\{G_\alpha \mid \alpha < \gamma\}$ be a discrete collection of cozero-sets of A and $\{F_\alpha \mid \alpha < \gamma\}$ be a collection of zero-sets of A with $F_\alpha \subset G_\alpha$ for every $\alpha < \gamma$. From the assumption, there exists a disjoint collection $\{G_\alpha^* \mid \alpha < \gamma\}$ of cozero-sets of X such that $\bigcup_{\alpha<\gamma} G_\alpha^*$ is a cozero-set of X and $G_\alpha^* \cap A = G_\alpha$ for every $\alpha < \gamma$. Let $f : X \to [0,1]$ be a continuous function such that $\bigcup_{\alpha<\gamma} G_\alpha^* = f^{-1}((0,1])$. Then, $\{G_\alpha^* \cap f^{-1}((1/n,1]) \mid \alpha < \gamma\}$ is a discrete collection of cozero-sets of X for each $n \in \mathbb{N}$ and $F_\alpha \subset \bigcup_{n\in\mathbb{N}}(G^*_\alpha \cap f^{-1}((1/n, 1]))$ for each $\alpha < \gamma$. Hence A is z_γ-embedded in X.

Because every two disjoint zero-sets of A are completely separated in X, A is C^*-embedded in X (e.g., [1, Theorem 6.6] or [6, p. 18]).

Next we shall show (5) $\Rightarrow$ (6). Assume that A is z_γ- and C^*-embedded in X. Let $f : A \to J(\gamma)$ be a continuous map and $\xi_1, \dots, \xi_n$ finitely many elements of γ. Let $U_\xi = f^{-1}((0, 1] \times \{\xi\})$ for every $\xi < \gamma$. Define continuous functions $f_i : A \to [0, 1]$ $(i = 1, \dots, n)$ by

$$f_i(a) = \begin{cases} (\Pi \circ f)(a) & \text{if } a \in U_{\xi_i}, \\ 0 & \text{otherwise} \end{cases}$$

for every $i = 1, \dots, n$. Then it follows that

$$f_i^{-1}\big((0, 1]\big) = U_{\xi_i} \tag{1}$$

for each $i = 1, \dots, n$. Since $\bigcup\{U_\xi \mid \xi < \gamma;\ \xi \neq \xi_1, \dots, \xi_n\}$ is a cozero-set of A, there exists a continuous function $k : A \to [0, 1]$ such that $\bigcup\{U_\xi \mid \xi < \gamma;\ \xi \neq \xi_1, \dots, \xi_n\} = k^{-1}((0, 1])$. Define a continuous map $h : A \to J(n + 1)$ by

$$h(a) = \begin{cases} (f_i(a),\ i) & \text{if } a \in U_{\xi_i} (i = 1, \dots, n), \\ (k(a),\ 0) & \text{if } a \in \bigcup\{U_\xi \mid \xi < \gamma;\ \xi \neq \xi_1, \dots, \xi_n\}, \\ \theta & \text{otherwise.} \end{cases}$$

Since $J(n + 1)$ is compact AR and A is C^*-embedded in X, there exists a continuous map $h^* : X \to J(n+1)$ such that $h^*|A = h$ (e.g., [9, Theorem 2.8(a)]). Since A is z_γ-embedded in X and $\{U_\xi \mid \xi < \gamma;\ \xi \neq \xi_1, \dots, \xi_n\}$ is a disjoint collection of cozero-sets of A whose union is a cozero-set of A, there exists a disjoint collection $\{V_\xi \mid \xi < \gamma;\ \xi \neq \xi_1, \dots, \xi_n\}$ of cozero-sets of X such that $\bigcup\{V_\xi \mid \xi < \gamma;\ \xi \neq \xi_1, \dots, \xi_n\}$ is a cozero-set of X and $V_\xi \cap A = U_\xi$ for each $\xi < \gamma$, $\xi \neq \xi_1, \dots, \xi_n$ because of (3) of Lemma 2.2. Put

$$V^*_\xi = V_\xi \cap h^{*-1}\big((0, 1] \times \{0\}\big)$$

for every $\xi < \gamma$, $\xi \neq \xi_1, \dots, \xi_n$. Since $\bigcup\{V^*_\xi \mid \xi < \gamma;\ \xi \neq \xi_1, \dots, \xi_n\}$ is a cozero-set of X, there exists a continuous function $\ell : X \to [0, 1]$ such that $\ell^{-1}((0, 1]) = \bigcup\{V^*_\xi \mid \xi < \gamma;\ \xi \neq \xi_1, \dots, \xi_n\}$. Define a map $g : X \to J(\gamma)$ by

$$g(x) = \begin{cases} ((\Pi \circ h^*)(x),\ \xi_i) & \text{if } x \in h^{*-1}((0, 1] \times \{i\})\ (i = 1, \dots, n), \\ (\ell(x),\ \xi) & \text{if } x \in V^*_\xi\ (\xi < \gamma,\ \xi \neq \xi_1, \dots, \xi_n), \\ \theta & \text{otherwise.} \end{cases}$$

We shall show that g is continuous. To see this, pick an $x \in X$ arbitrarily and let $\varepsilon > 0$ be arbitrarily. It suffices to show for the case

$$x \notin \bigcup_{i=1}^{n} h^{*-1}\big((0, 1] \times \{i\}\big) \cup \bigcup\big\{V^*_\xi \mid \xi < \gamma;\ \xi \neq \xi_1, \dots, \xi_n\big\}.$$

Since $\ell(x)=0$ and ℓ is continuous, there exists a neighborhood O_x of x in X such that $\ell(O_x)\subset[0,\varepsilon)$. We first assume $h^*(x)=\theta$. Then there exists a neighborhood O'_x of x such that

$$h^*(O'_x)\subset\bigcup_{i=0}^{n}\big([0,\varepsilon)\times\{i\}\big).$$

Then we can easily show that $\rho(g(x),g(y))<\varepsilon$ for each $y\in O_x\cap O'_x$. Next we assume $h^*(x)\in(0,1]\times\{0\}$. Then we have $\rho(g(x),g(y))<\varepsilon$ for each $y\in O_x\cap h^{*-1}((0,1]\times\{0\})$. Hence g is continuous.

To prove (i), pick an $a\in f^{-1}((0,1]\times\{\xi_i\})$ for some $i=1,\dots,n$. Since $a\in f^{-1}((0,1]\times\{\xi_i\})=U_{\xi_i}$, we have

$$f(a)=\big((\Pi\circ f)(a),\ \xi_i\big)=\big(f_i(a),\ \xi_i\big). \tag{2}$$

Moreover we have

$$h^*(a)=h(a)=\big(f_i(a),\ i\big). \tag{3}$$

It follows from the fact $h^*(a)\in(0,1]\times\{i\}$ that $g(a)=((\Pi\circ h^*)(a),\ \xi_i)$. By (3), we have

$$(\Pi\circ h^*)(a)=f_i(a). \tag{4}$$

Therefore it follows from (2) and (4) that

$$g(a)=\big((\Pi\circ h^*)(a),\ \xi_i\big)=\big(f_i(a),\ \xi_i\big)=f(a).$$

So (i) is satisfied.

To prove (ii), it suffices to show the case $\xi<\gamma$, $\xi\neq\xi_1,\dots,\xi_n$, because other cases follow from (i) and the fact that $g^{-1}((0,1]\times\{\xi_i\})=(h^*)^{-1}((0,1]\times\{i\})$ for every $i=1,\dots,n$. Fix a $\xi<\gamma$, $\xi\neq\xi_1,\dots,\xi_n$. First, we shall prove that $U_\xi\subset g^{-1}((0,1]\times\{\xi\})\cap A$. Pick an $a\in U_\xi$ be arbitrarily. Since $a\in U_\xi=V_\xi\cap A$, and $h^*(a)=h(a)=(k(a),0)\subset(0,1]\times\{0\}$, we have $a\in V^*_\xi$ and $\ell(a)\in(0,1]$. It follows that $g(a)=(\ell(a),\ \xi)\subset(0,1]\times\{\xi\}$. So we have $U_\xi\subset g^{-1}((0,1]\times\{\xi\})\cap A$. Next, to prove $g^{-1}((0,1]\times\{\xi\})\cap A\subset U_\xi$, pick an $a\in g^{-1}((0,1]\times\{\xi\})\cap A$. Since $(j\circ g)(a)=\xi$, we have $a\in V^*_\xi\cap A\subset V_\xi\cap A=U_\xi$ by the definition of g. Hence $g^{-1}((0,1]\times\{\xi\})\cap A\subset U_\xi$. So, (ii) is satisfied. Thus (6) follows.

We shall prove (6) $\Rightarrow$ (1). Let $\{U_\alpha\mid\alpha<\gamma\}$ be a disjoint collection of cozero-sets of A satisfying $\bigcup\{U_\alpha\mid\alpha<\gamma\}$ is a cozero-set of A and any finitely many $\alpha_1,\dots,\alpha_n$'s $<\gamma$ satisfying $U_{\alpha_i}=f_{\alpha_i}^{-1}((0,1])$ for some continuous functions $f_{\alpha_i}:A\to[0,1]$ $(i=1,\dots,n)$. Let $h:A\to[0,1]$ be a continuous function such that $h^{-1}((0,1])=\bigcup\{U_\alpha\mid\alpha<\gamma;\ \alpha\neq\alpha_1,\dots,\alpha_n\}$. Define a continuous map $f:A\to J(\gamma)$ by

$$f(a)=\begin{cases}(f_{\alpha_i}(a),\ \alpha_i) & \text{if } a\in U_{\alpha_i}\ (i=1,\dots,n),\\ (h(a),\ \alpha) & \text{if } a\in U_\alpha\ (\alpha<\gamma,\ \alpha\neq\alpha_1,\dots,\alpha_n),\\ \theta & \text{otherwise.}\end{cases}$$

From assumption, there exists a continuous map $g : X \to J(\gamma)$ such that the conditions (i) and (ii) are satisfied. Define, for each $\alpha < \gamma$, $U^*_\alpha = g^{-1}((0,1] \times \{\alpha\})$. Moreover, define continuous functions $f^*_{\alpha_i} : X \to [0,1]$ $(i = 1, \ldots, n)$ by

$$f^*_{\alpha_i}(x) = \begin{cases} (\Pi \circ g)(x) & \text{if } x \in U^*_{\alpha_i}, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\{U^*_\alpha \mid \alpha < \gamma\}$ is a collection of cozero-sets of X and the conditions (a), (b) and (c) are satisfied.

(6) $\Rightarrow$ (7) is easy to show. The proof of (7) $\Rightarrow$ (2) is similar to that of (6) $\Rightarrow$ (1). The proof is completed. $\square$

Remark 3.3. In the definition of $(P^*)^\gamma$-embedding or (2) of Theorem 3.2, we can change "a disjoint collection $\{U_\alpha \mid \alpha < \gamma\}$ of cozero-sets of A satisfying $\bigcup\{U_\alpha \mid \alpha < \gamma\}$ is a cozero-set of A" into "a discrete collection $\{U_\alpha \mid \alpha < \gamma\}$ of cozero-sets of A". Similarly, in (3) or (4) of Theorem 3.2, we can change "a discrete collection $\{U_\alpha \mid \alpha < \gamma\}$ of cozero-sets of A" into "a disjoint collection $\{U_\alpha \mid \alpha < \gamma\}$ of cozero-sets of A satisfying $\bigcup\{U_\alpha \mid \alpha < \gamma\}$ is a cozero-set of A".

Since "$z_\omega + C^* = C^*$", by (5) of Theorem 3.2, we have the following:

Corollary 3.4. *A subspace A of a space X is C^*-embedded in X if and only if A is $(P^*)^\omega$-embedded in X.*

Since (3) of Proposition 2.1 implies (6) of Theorem 3.2, we have the following:

Proposition 3.5. *If A is a P^γ-embedded subspace of X, then A is $(P^*)^\gamma$-embedded in X.*

Remark 3.6. There exists a space with a subspace which is $(P^*)^\gamma$-embedded but not P^γ-embedded. For example, in Katětov's $\Gamma = \beta\mathbb{R} - (\beta\mathbb{N} - \mathbb{N})$ [11], $\mathbb{N}$ is C^*- and z_γ-embedded but not C-embedded in Γ. Easy examples show that $(P^*)^\gamma$-embedding need not be implied by z_γ-embedding.

By Hoshina [7], a subspace A of a space X is said to be *U^γ-embedded* in X if every uniformly locally finite collection $\mathcal{U}$ of subsets of A with $|\mathcal{U}| \leqslant \gamma$ is uniformly locally finite in X. Since P^γ-embedding implies U^γ-embedding [7, p. 52] and U^ω-embedding implies well-embedding ([7, Lemma 1.5] and [10]), we have the following corollary; with Corollary 3.4 this answers to Ohta's problem mentioned in the introduction.

Corollary 3.7. *Let X be a space and A a subspace. A is P^γ-embedded in X if and only if A is $(P^*)^\gamma$- and U^ω-embedded in X.*

4. Approximations of continuous maps with hedgehog-values

In this section, we study two types of approximations of continuous maps with hedgehog-values, where approximation means continuous maps on a subspace A of X into $J(\gamma)$ are approximated (on A) by continuous maps on X into $J(\gamma)$. First, with this notion we give another characterization of $(P^*)^\gamma$-embedding.

Theorem 4.1. *Let X be a space and A a subspace. The following statements are equivalent*:

(1) *A is $(P^*)^\gamma$-embedded in X*;

(2) *for every continuous map $f : A \to J(\gamma)$, any finitely many $\xi_1, \dots, \xi_n$'s $< \gamma$ and any $\varepsilon > 0$, there exists a continuous map $g : X \to J(\gamma)$ such that the following conditions are satisfied*:
 (i)′ *$\rho(g(a), f(a)) < \varepsilon$ for each $a \in f^{-1}(\bigcup_{i=1}^n ((0,1] \times \{\xi_i\}))$, and*
 (ii) *$g^{-1}((0,1] \times \{\xi\}) \cap A = f^{-1}((0,1] \times \{\xi\})$ for each $\xi < \gamma$*;

(3) *for every continuous map $f : A \to J(\gamma)$, any $\xi_1 < \gamma$ and any $\varepsilon > 0$ there exists a continuous map $g : X \to J(\gamma)$ such that* (i)′ *and* (ii) *of* (2) *are satisfied.*

Proof. (1) ⇒ (2) follows from (6) of Theorem 3.2. (2) ⇒ (3) is obvious. To prove (3) ⇒ (1), we assume (3) is satisfied. We shall show that (5) of Theorem 3.2 is satisfied. To prove C^*-embedding of A, let Z_0, Z_1 be disjoint zero-sets of A. Let $k : A \to [1/4, 1]$ be a continuous function such that $k^{-1}(\{1/4\}) = Z_0$ and $k^{-1}(\{1\}) = Z_1$. Let $i : [1/4, 1] \to J(\gamma)$ be the inclusion map so that $i(r) = (r, 0)$ for every $r \in [1/4, 1]$. By the assumption, there exists a continuous map $g : X \to J(\gamma)$ such that $\rho(g(a), (i \circ k)(a)) < 1/4$ for every $a \in (i \circ k)^{-1}((0,1] \times \{0\}) = A$. We have that $Z_0 \subset g^{-1}([0, 1/2] \times \{0\})$ and $Z_1 \subset g^{-1}([3/4, 1] \times \{0\})$. Since $g^{-1}([0, 1/2] \times \{0\})$ and $g^{-1}([3/4, 1] \times \{0\})$ are disjoint zero-sets in X, it follows that Z_0 and Z_1 are completely separated in X. Hence A is C^*-embedded in X. And it follows from (4) of Lemma 2.2 that A is z_γ-embedded in X. Hence A is $(P^*)^\gamma$-embedded in X, it completes the proof. □

Theorem 4.1 suggests a characterization of P^γ-embedding in terms of approximations of continuous maps into $J(\gamma)$.

Theorem 4.2. *A subspace A of a space X is P^γ-embedded in X if and only if for every continuous map $f : A \to J(\gamma)$, any subset Ω of γ with $|\Omega| \leqslant \omega$ and any $\varepsilon > 0$ there exists a continuous map $g : X \to J(\gamma)$ such that the following two conditions are satisfied*:
 (i)′ *$\rho(g(a), f(a)) < \varepsilon$ for each $a \in f^{-1}(\bigcup_{\xi \in \Omega} ((0,1] \times \{\xi\}))$, and*
 (ii) *$g^{-1}((0,1] \times \{\xi\}) \cap A = f^{-1}((0,1] \times \{\xi\})$ for each $\xi < \gamma$.*

To prove this, we need a lemma.

Lemma 4.3. *Let X be a space and A a subspace. Assume that for every continuous map $f : A \to J(\omega)$ and any $\varepsilon > 0$, there exists a continuous map $g : X \to J(\omega)$ such that $\rho(g(a), f(a)) < \varepsilon$ for each $a \in A$. Then A is well-embedded in X.*

Proof. Let Z be a zero-set of X disjoint from A. It suffices to show that there exists a zero-set of X which contains A and is disjoint from Z. Let $f : X \to [0,1]$ be a continuous function such that $Z = f^{-1}(\{0\})$. Let

$$\mathcal{G}_1 = \left\{ f^{-1}\left(\left(\frac{1}{4n+2}, \frac{1}{4n-1}\right)\right) \cap A \;\middle|\; n \in \mathbb{N} \right\}$$

and

$$\mathcal{F}_1 = \left\{ f^{-1}\left(\left[\frac{1}{4n+1}, \frac{1}{4n}\right]\right) \cap A \;\middle|\; n \in \mathbb{N} \right\}.$$

Since $\bigcup \mathcal{G}_1$ is a cozero-set of A and $\bigcup \mathcal{F}_1$ is a zero-set of A, there exists a continuous function $k : A \to [0,1]$ such that $k^{-1}(\{1\}) = \bigcup \mathcal{F}_1$ and $k^{-1}((0,1]) = \bigcup \mathcal{G}_1$. Define a continuous map $h : A \to J(\omega)$ such that

$$h(a) = \begin{cases} (k(a), n) & \text{if } a \in f^{-1}\left(\left(\dfrac{1}{4n+2}, \dfrac{1}{4n-1}\right)\right) \cap A \ (n \in \mathbb{N}), \\ \theta & \text{otherwise.} \end{cases}$$

From the assumption, there exists a continuous map $g : X \to J(\omega)$ such that $\rho(g(a), h(a)) < 1/2$ for each $a \in A$. Then we have $f^{-1}([1/(4n+1), 1/(4n)]) \cap A \subset g^{-1}([1/2, 1] \times \{n\})$ for each $n \in \mathbb{N}$. Put

$$Z_n^1 = f^{-1}\left(\left[\frac{1}{4n+1}, \frac{1}{4n}\right]\right) \cap g^{-1}\left(\left[\frac{1}{2}, 1\right] \times \{n\}\right)$$

for every $n \in \mathbb{N}$. Then $\mathcal{Z}_1 = \{Z_n^1 \mid n < \omega\}$ is a discrete collection of zero-sets of X which has a discrete expansion of cozero-sets of X.

In the same way, we also take a discrete collection $\mathcal{Z}_i$ of zero-sets of X which has a discrete expansion of cozero-sets of X corresponding to the discrete collection

$$\mathcal{G}_i = \left\{ f^{-1}\left(\left(\frac{1}{4n+3-i}, \frac{1}{4n-i}\right)\right) \cap A \;\middle|\; n \in \mathbb{N} \right\}$$

of cozero-sets of A and a collection

$$\mathcal{F}_i = \left\{ f^{-1}\left(\left[\frac{1}{4n+2-i}, \frac{1}{4n+1-i}\right]\right) \cap A \;\middle|\; n \in \mathbb{N} \right\}$$

of zero-sets of A, where $i = 2, 3$ and 4 (in particular, let $f^{-1}((1/(4n+3-i), 1/(4n-i))) = f^{-1}((1/(4n+3-i), 1])$ in the case of $i = 4$ and $n = 1$). Then, $\bigcup\{Z_n^i \mid n \in \mathbb{N}, i = 1, \ldots, 4\}$ is a zero-set of X which contains A and is disjoint from Z. Thus A is well-embedded in X, it completes the proof. $\square$

Proof of Theorem 4.2. We only prove the "if" part, because the "only if" part easily follows from (3) of Proposition 2.1. To prove the "if" part, we assume the condition of the "if" part of Theorem 4.2 to be satisfied. We first show that the assumption of Lemma 4.3 is satisfied. To see this, let $f : A \to J(\omega)$ be a continuous map and $\varepsilon > 0$. Let $i : J(\omega) \to J(\gamma)$ be the inclusion map such as $i(x) = (\Pi(x),\ j(x))$ if $\Pi(x) > 0$ and $i(\theta) = \theta$. From the assumption of Theorem 4.2, there exists a continuous map $k : X \to J(\gamma)$ such that $\rho(k(a), (i \circ f)(a)) < \varepsilon$ for each $a \in (i \circ f)^{-1}(\bigcup_{\xi<\omega}((0,1] \times \{\xi\}))$

and $k^{-1}((0,1]\times\{\xi\})\cap A=(i\circ f)^{-1}((0,1]\times\{\xi\})$ for each $\xi<\gamma$. Define a continuous map $h: J(\gamma)\to J(\omega)$ by

$$h(y)=\begin{cases}(\Pi(y),\ j(y)) & \text{if } j(y)<\omega,\\ \theta & \text{otherwise.}\end{cases}$$

Let $g=h\circ k$. Then $\rho(g(a),f(a))<\varepsilon$ for each $a\in A$. Hence $g: X\to J(\omega)$ is the required map satisfying the assumption of Lemma 4.3. It follows from Lemma 4.3 that A is well-embedded in X. On the other hand, A is z_γ-embedded in X because of (3) of Lemma 2.2. Hence A is P^γ-embedded in X. □

In view of conditions (i)′ of Theorem 4.1(2) and (i)′ of Theorem 4.2, we see that the difference between P^γ and $(P^*)^\gamma$-embedding only depends on the number of spines on which continuous maps can be approximated. The following Corollary 4.4 immediately follows from Proposition 2.1 and Theorem 4.2. This result and (6) of Theorem 3.2 show that the difference of these extension properties also appears on the number of spines on which continuous maps can be extended.

Corollary 4.4. *A subspace A of a space X is P^γ-embedded in X if and only if for every continuous map $f: A\to J(\gamma)$ and any subset Ω of γ with $|\Omega|\leqslant\omega$ there exists a continuous map $g: X\to J(\gamma)$ such that the following two conditions are satisfied*:

(i) $g(a)=f(a)$ *for each* $a\in f^{-1}(\bigcup_{\xi\in\Omega}((0,1]\times\{\xi\}))$, *and*

(ii) $g^{-1}((0,1]\times\{\xi\})\cap A=f^{-1}((0,1]\times\{\xi\})$ *for each* $\xi<\gamma$.

Remark 4.5. Lemma 4.3 cannot be reversed. Because, in Corollary 4.8, C-embedding of A will be implied from the assumption of Lemma 4.3.

Next we characterize P^γ- or z_γ-embedding by another type of approximation of the hedgehog-valued continuous maps. At first we give a technical lemma.

Lemma 4.6. *Let X be a space and A a subspace. Assume that for any continuous map $f: A\to J(\gamma)$ there exists a cozero-set G of X containing A and a continuous map $g: G\to J(\gamma)$ such that $\rho(g(a),f(a))<1/3$ for each $a\in A$. Then for any discrete collection $\{G_\alpha\mid\alpha<\gamma\}$ of cozero-sets of A and a collection $\{F_\alpha\mid\alpha<\gamma\}$ of zero-sets of A satisfying $F_\alpha\subset G_\alpha$ for every $\alpha<\gamma$, there exist a cozero-set H of X containing A, a discrete collection $\{G^*_\alpha\mid\alpha<\gamma\}$ of cozero-sets of H and a collection $\{F^*_\alpha\mid\alpha<\gamma\}$ of zero-sets of H such that $F_\alpha\subset F^*_\alpha\subset G^*_\alpha$ and $G^*_\alpha\cap A\subset G_\alpha$ for every $\alpha<\gamma$. Moreover if in addition we always assume $G=X$ for every f, then we can take $H=X$.*

Proof. Let $\{G_\alpha\mid\alpha<\gamma\}$ be a discrete collection of cozero-sets of A and $\{F_\alpha\mid\alpha<\gamma\}$ be a collection of zero-sets of A such that $F_\alpha\subset G_\alpha$ for every $\alpha<\gamma$. Since $\bigcup_{\alpha<\gamma}G_\alpha$ is a cozero-set of A and $\bigcup_{\alpha<\gamma}F_\alpha$ is a zero-set of A, there exists a continuous function

$h : A \to [0,1]$ such that $h^{-1}(\{1\}) = \bigcup_{\alpha<\gamma} F_\alpha$ and $h^{-1}((0,1]) = \bigcup_{\alpha<\gamma} G_\alpha$. Define a continuous map $f : A \to J(\gamma)$ by

$$f(a) = \begin{cases} (h(a), \alpha) & \text{if } a \in G_\alpha \ (\alpha < \gamma), \\ \theta & \text{otherwise.} \end{cases}$$

From the assumption, there exists a cozero-set G of X containing A and a continuous map $g : G \to J(\gamma)$ such that $\rho(g(a), f(a)) < 1/3$ for each $a \in A$. Let $G^*_\alpha = g^{-1}((1/3, 1] \times \{\alpha\})$ and $F^*_\alpha = g^{-1}([2/3, 1] \times \{\alpha\})$ for every $\alpha < \gamma$. Then, $\{G^*_\alpha \mid \alpha < \gamma\}$ and $\{F^*_\alpha \mid \alpha < \gamma\}$ are the required collections. The last statement is easily seen. The proof is completed. □

Theorem 4.7. *A subspace A of a space X is P^γ-embedded in X if and only if for every continuous map $f : A \to J(\gamma)$ and any $\varepsilon > 0$ there exists a continuous map $g : X \to J(\gamma)$ such that $\rho(g(a), f(a)) < \varepsilon$ for every $a \in A$.*

Proof. We only prove the "if" part, because the "only if" part follows from (3) of Proposition 2.1. We assume the condition of the "if" part is satisfied. Let $\{U_\alpha \mid \alpha < \gamma\}$ be a uniformly discrete collection of subsets of A. By Blair [3, Theorem 4.6], to prove P^γ-embedding of A, it suffices to show that $\{U_\alpha \mid \alpha < \gamma\}$ is uniformly discrete also in X. Take a discrete collection $\{G_\alpha \mid \alpha < \gamma\}$ of cozero-sets of A and a collection $\{F_\alpha \mid \alpha < \gamma\}$ of zero-sets of A such that $U_\alpha \subset F_\alpha \subset G_\alpha$ for every $\alpha < \gamma$. By Lemma 4.6, there exists a discrete collection $\{G^*_\alpha \mid \alpha < \gamma\}$ of cozero-sets of X and a collection $\{F^*_\alpha \mid \alpha < \gamma\}$ of zero-sets of X such that $F_\alpha \subset F^*_\alpha \subset G^*_\alpha$ for every $\alpha < \gamma$. Thus, $\{U_\alpha \mid \alpha < \gamma\}$ is a uniformly discrete collection of X. Hence A is P^γ-embedded in X. It completes the proof. □

Corollary 4.8. *A subspace A of a space X is C-embedded in X if and only if for every continuous map $f : A \to J(\omega)$ and any $\varepsilon > 0$ there exists a continuous map $g : X \to J(\omega)$ such that $\rho(g(a), f(a)) < \varepsilon$ for every $a \in A$.*

In [4, Theorem 2.2] Blair and Hager proved that a subspace A of a space X is z-embedded in X if and only if for any continuous function $f : A \to [0,1]$ and any $\varepsilon > 0$ there exist a cozero-set G of X containing A and a continuous function $g : G \to [0,1]$ such that $|g(a) - f(a)| < \varepsilon$ for each $a \in A$. Theorem 4.9 below extends this result to the case of z_γ-embedding; (1) ⇔ (2) was stated by Tsukada [19], however his proof is incomplete. Here we give our proof using the technique in Blair and Hager [4].

Theorem 4.9. *Let X be a space and A a subspace. The following statements are equivalent*:

(1) *A is z_γ-embedded in X;*
(2) *for every continuous map $f : A \to J(\gamma)$ and any $\varepsilon > 0$ there exist a cozero-set G of X containing A and a continuous map $g : G \to J(\gamma)$ such that $\rho(g(a), f(a)) < \varepsilon$ for every $a \in A$;*
(3) *for every continuous map $f : A \to (J(\gamma)^\omega, \rho')$ and any $\varepsilon > 0$, there exists a cozero-set G containing A and a continuous map $g : G \to (J(\gamma)^\omega, \rho')$ such that*

$\rho'(g(a), f(a)) < \varepsilon$ for each $a \in A$, where ρ' is the metric defined by $\rho'(x, y) = \sum_{i=1}^{\infty} 2^{-i}\rho(x_i, y_i)$ for every $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots) \in J(\gamma)^{\omega}$.

Proof. To prove (1) $\Rightarrow$ (2), assume A is z_γ-embedded in X. Let $f : A \to J(\gamma)$ be a continuous map and $\varepsilon > 0$. We can assume $\varepsilon \leqslant 1$. Take a positive integer n such that $n\varepsilon \leqslant 1 < (n+1)\varepsilon$. For every $\xi < \gamma$, we put

$$U_\xi^1 = f^{-1}\big((0,\ 2\varepsilon) \times \{\xi\}\big),$$

$$U_\xi^i = f^{-1}\Big(\big((i-1)\varepsilon,\ (i+1)\varepsilon\big) \times \{\xi\}\Big)$$

for $i = 2, \ldots, n-1$ and

$$U_\xi^n = f^{-1}\Big(\big((n-1)\varepsilon,\ 1\big] \times \{\xi\}\Big).$$

Since $\{f^{-1}((0, 1] \times \{\xi\}) \mid \xi < \gamma\}$ is a disjoint collection of cozero-sets of A and whose union is a cozero-set of A, from (3) of Lemma 2.2, there exists a disjoint collection $\{W_\xi^* \mid \xi < \gamma\}$ of cozero-sets of X such that $\bigcup\{W_\xi^* \mid \xi < \gamma\}$ is a cozero-set of X and $W_\xi^* \cap A = f^{-1}((0, 1] \times \{\xi\})$ for each $\xi < \gamma$. Since A is z-embedded in X, there exists a cozero-set U_ξ^{i*} of X such that $U_\xi^{i*} \cap A = U_\xi^i$ for every $\xi < \gamma$ and $i = 1, \ldots, n$. Similarly, there exists a cozero-set H_0 of X such that

$$H_0 \cap A = f^{-1}\left(\bigcup_{\xi<\gamma}\big([0, \varepsilon) \times \{\xi\}\big)\right).$$

Let $H_\xi^i = U_\xi^{i*} \cap W_\xi^*$ for each $\xi < \gamma$ and $i = 1, \ldots, n$.

Here we shall show that $\bigcup_{\xi<\gamma} H_\xi^i$ is a cozero-set of X for every $i = 1, \ldots, n$. Since H_ξ^i is a cozero-set of X, it is a cozero-set of W_ξ^*. Then, $\bigcup_{\xi<\gamma} H_\xi^i$ is a cozero-set of $\bigcup_{\xi<\gamma} W_\xi^*$, because $\{W_\xi^* \mid \xi < \gamma\}$ is a disjoint open collection. Since $\bigcup_{\xi<\gamma} W_\xi^*$ is a cozero-set of X, it follows that $\bigcup_{\xi<\gamma} H_\xi^i$ is a cozero-set of X.

Let $g_i : A \to [0, 1]$ $(i = 0, 1, \ldots, n)$ be continuous functions such that $H_0 = g_0^{-1}((0, 1])$ and $\bigcup_{\xi<\gamma} H_\xi^i = g_i^{-1}((0, 1])$. Put $G = H_0 \cup \bigcup\{H_\xi^i \mid \xi < \gamma;\ i = 1, \ldots, n\}$. Then it is clear that G is a cozero-set of X and contains A. Since $g_0 + g_1 + \cdots + g_n$ is positive on G, we can define $g_i' : G \to [0, 1]$ by

$$g_i' = \frac{g_i}{g_0 + g_1 + \cdots + g_n}$$

for every $i = 0, 1, \ldots, n$. Then, $\sum_{i=0}^n g_i' = 1$ and g_i' $(i = 0, 1, \ldots, n)$ are continuous (on G). Consequently we define a continuous function $k : G \to [0, 1]$ by $k = \varepsilon(\sum_{i=1}^n i \cdot g_i')$. The required map $g : G \to J(\gamma)$ can be defined by

$$g(x) = \begin{cases} (k(x),\ \xi) & \text{if } x \in \bigcup_{i=1}^n H_\xi^i\ (\xi < \gamma), \\ \theta & \text{otherwise.} \end{cases}$$

Just as in Blair and Hager [4], it follows that g is continuous and $\rho(g(a), f(a)) < \varepsilon$ for each $a \in A$, which proves (1) $\Rightarrow$ (2).

To prove (2) $\Rightarrow$ (3), we assume (2) is satisfied. Let $f : A \to J(\gamma)^\omega$ be a continuous map and any $\varepsilon > 0$. Let an $n \in \mathbb{N}$ such that $2^{n-1}\varepsilon > 1$. From the assumption, for each

$i = 1, \ldots, n$, there exist a cozero-set G_i contains A and a continuous map $g_i : G_i \to J(\gamma)$ such that $\rho(g_i(a), (p_i \circ f)(a)) < \varepsilon/n$ for each $a \in A$, where p_i is the projection from $J(\gamma)^\omega$ into the ith coordinate $J(\gamma)$. Define a map $g : \bigcap_{i=1}^n G_i \to J(\gamma)^\omega$ such that

$$g(x) = \big(g_1(x), \ldots, g_n(x), \theta, \theta, \ldots\big)$$

for each $x \in \bigcap_{i=1}^n G_i$. Then it is easy to show that g is continuous and $\rho(g(a), f(a)) < \varepsilon$ for each $a \in A$, which proves (2) $\Rightarrow$ (3).

To prove (3) $\Rightarrow$ (1), assume (3) is satisfied. Let $f : A \to J(\gamma)$ be a continuous map. We first show that there exists a cozero-set G of X containing A and a continuous map $g : G \to J(\gamma)$ such that $\rho(g(a), f(a)) < 1/3$ for each $a \in A$. Define an inclusion map $i : J(\gamma) \to J(\gamma)^\omega$ by $i(y) = (y, \theta, \theta, \ldots)$ for every $y \in J(\gamma)$. From the assumption, there exist a cozero-set G of X containing A and a continuous map $g : G \to J(\gamma)^\omega$ such that $\rho'(g(a), (i \circ f)(a)) < 1/6$ for every $a \in A$. Let p_1 be the projection map from $J(\gamma)^\omega$ to the first coordinate $J(\gamma)$, i.e., $p_1(y) = y_1$ for $y = (y_i) \in J(\gamma)^\omega$. Then the map $p_1 \circ g : G \to J(\gamma)$ satisfies $\rho((p_1 \circ g)(a), f(a)) < 1/3$ for every $a \in A$. To show that (2) of Lemma 2.2, let $\{G_\alpha \mid \alpha < \gamma\}$ be a discrete collection of cozero-sets of A and $\{F_\alpha \mid \alpha < \gamma\}$ a collection of zero-sets of A with $F_\alpha \subset G_\alpha$ for each $\alpha < \gamma$. From Lemma 4.6, there exists a cozero-set H of X containing A and a discrete collection $\{G^*_\alpha \mid \alpha < \gamma\}$ of cozero-sets of H such that $F_\alpha \subset G^*_\alpha$ and $G^*_\alpha \cap A \subset G_\alpha$ for every $\alpha < \gamma$. Let $h : X \to [0, 1]$ be a continuous functions such that $h^{-1}((0, 1]) = H$. Let $H_{i\alpha} = G^*_\alpha \cap h^{-1}((1/i,\ 1])$ for every $\alpha < \gamma$ and $i \in \mathbb{N}$. It is easy to see that $\mathcal{H}_i = \{H_{i\alpha} \mid \alpha < \gamma\}$ is a discrete collection of cozero-sets of X satisfying that $H_{i\alpha} \cap A \subset G_\alpha$ and $F_\alpha \subset \bigcup_{i \in \mathbb{N}} H_{i\alpha}$ for each $\alpha < \gamma$. From (2) of Lemma 2.2, A is z_γ-embedded in X. It completes the proof. $\square$

As is known, every Čech-complete metric space Y with weight $Y \leqslant \gamma$ can be homeomorphically embedded in $J(\gamma)^\omega$ as a closed subset (e.g., [5]). Hence, in view of Theorem 4.9, the following problem naturally arises. A similar problem is stated in [19]. Note that the corresponding case of P^γ-embedding is positive by Przymusiński [17] (see (4) of Proposition 2.1).

Problem 4.10. Can "$J(\gamma)$" be changed into "any Čech-complete AR space with weight $\leqslant \gamma$ for some (or any) metric ρ'" in (2) of Theorem 4.9?

Remark 4.11. Theorem 4.7 or Corollary 4.8 can also be shown by Lemma 4.3 and Theorem 4.9.

5. Weak z_γ-embedding

In the arguments in preceding sections one may naturally ask, for a subspace A of a space X, what condition plus z-embedding is equal to z_γ-embedding. Such a condition seems to have been unknown. We can easily see that such a condition plus C^*-embedding (respectively C-embedding) would be equal to $(P^*)^\gamma$-embedding (respectively

P^γ-embedding). In [14] Morita and Hoshina proved a related result that A is P^γ-embedded in X if and only if A is C-embedded in X and for every discrete collection $\{F_\alpha \mid \alpha < \gamma\}$ of zero-sets of A having a discrete expansion $\{G_\alpha \mid \alpha < \gamma\}$ of cozero-sets of A there exists a locally finite collection $\{H_\alpha \mid \alpha < \gamma\}$ of cozero-sets of X such that $F_\alpha \subset H_\alpha \cap A \subset G_\alpha$ for each $\alpha < \gamma$. It is easy to show that the latter condition implies A being z_γ-embedded in X. Extending this condition, in view of (2) of Lemma 2.2, we define weak z_γ-embedding as follows.

Let X be a space and A a subspace. We say that A is *weakly z_γ-embedded* in X if for any uniformly discrete collection $\{F_\alpha \mid \alpha < \gamma\}$ of zero-sets of A (i.e., discrete collection $\{F_\alpha \mid \alpha < \gamma\}$ of zero-sets of A that has a discrete expansion of cozero-sets of A), for each $i < \omega$ there exists a locally finite collection $\mathcal{H}_i = \{H_{i\alpha} \mid \alpha < \gamma\}$ of cozero-sets of X such that $F_\alpha \subset \bigcup\{H_{i\alpha} \mid i < \omega\}$ for each $\alpha < \gamma$. If A is weakly z_γ-embedded in X for every γ, A is said to be *weakly z_∞-embedded* in X. Note that any subspace of any space is weakly z_ω-embedded. Since (2) of Lemma 2.2 implies weak z_γ-embedding, any z_γ-embedded subspace is weakly z_γ-embedded.

We now prove the following proposition which was stated in the introduction. (1) of the following proposition gives an answer to the above question. In [7, Theorem 1.7] Hoshina actually proved that A is U^γ-embedded in X if and only if A is U^ω-embedded in X and every uniformly discrete collection $\mathcal{U}$ of zero-sets of A with $|\mathcal{U}| \leqslant \gamma$ is uniformly locally finite in X; the latter condition trivially implies weak z_γ-embedding of A over X. Hence (4) of the following proposition extends this result.

Proposition 5.1. *Let X be a space and A a subspace.*

(1) *A is z_γ-embedded in X if and only if A is z- and weakly z_γ-embedded in X.*
(2) *A is $(P^*)^\gamma$-embedded in X if and only if A is C^*- and weakly z_γ-embedded in X.*
(3) *A is P^γ-embedded in X if and only if A is C- and weakly z_γ-embedded in X.*
(4) *A is U^γ-embedded in X if and only if A is U^ω- and weakly z_γ-embedded in X.*

To prove this, we need a lemma; the proof is easy and omitted.

Lemma 5.2. *A subspace A of a space X is weakly z_γ-embedded in X if and only if for any uniformly discrete collection $\{F_\alpha \mid \alpha < \gamma\}$ of zero-sets of A, there exist a locally finite collection $\{H_{i\alpha} \mid \alpha < \gamma\}$ of cozero-sets of X, a collection $\{Z_{i\alpha} \mid \alpha < \gamma\}$ of zero-sets of A and a collection $\{E_{i\alpha} \mid \alpha < \gamma\}$ of cozero-sets of A for each $i < \omega$ such that the following conditions are satisfied*:

(a) *$E_{i\alpha} \subset Z_{i\alpha} \subset H_{i\alpha}$ for each $\alpha < \gamma$ and $i < \omega$, and*
(b) *$F_\alpha \subset \bigcup\{E_{i\alpha} \mid i < \omega\}$ for each $\alpha < \gamma$.*

Proof of Proposition 5.1. We easily verify (1) is satisfied by (2) of Lemma 2.2. (2) follows from (5) of Theorem 3.2 and (1) of this proposition. As was commented in Section 2, since A is C- (respectively P^γ-) embedded in X if and only if A is z- (respectively z_γ-) embedded and well-embedded in X, (3) also follows from (1) of this proposition.

Let us prove (4). From the definitions, obviously, U^γ-embedding implies weak z_γ-embedding. Hence, the "only if" part is clear. To prove the "if" part, we assume A is U^ω- and weak z_γ-embedded in X. Let $\{F_\alpha \mid \alpha < \gamma\}$ be a uniformly discrete collection of zero-sets of A. To prove U^γ-embedding of A, it suffices to show that $\{F_\alpha \mid \alpha < \gamma\}$ is uniformly locally finite in X. Since A is weakly z_γ-embedded in X, there exists a locally finite collection $\{H_{i\alpha} \mid \alpha < \gamma\}$ of cozero-sets of X, a collection $\{Z_{i\alpha} \mid \alpha < \gamma\}$ of zero-sets of A and a collection $\{E_{i\alpha} \mid \alpha < \gamma\}$ of cozero-sets of A for each $i < \omega$ such that the conditions (a) and (b) of Lemma 5.2 are satisfied. For every $n < \omega$, let

$$U_n = \Bigl(A - \bigcup\{F_\alpha \mid \alpha < \gamma\}\Bigr) \cup \Biggl(\bigcup_{i\leqslant n}\bigcup_{\alpha<\gamma}\Bigl(E_{i\alpha} \cap A - \bigcup\{F_\beta \mid \beta < \gamma;\ \beta \neq \alpha\}\Bigr)\Biggr).$$

Then, $\{U_n \mid n < \omega\}$ is an increasing cover of cozero-sets of A. Since A is U^ω-embedded in X, by [7, Corollary 1.4], there exist a locally finite cover $\{W_n \mid n < \omega\}$ of cozero-sets of X and a cover $\{V_n \mid n < \omega\}$ of zero-sets of X such that $V_n \subset W_n$ and $W_n \cap A \subset U_n$ for every $n < \omega$. Note that $\{W_n \cap H_{i\alpha} \mid i \leqslant n,\ \alpha < \gamma,\ n < \omega\}$ is a locally finite collection of cozero-sets of X and that $\{V_n \cap Z_{i\alpha} \mid i \leqslant n,\ \alpha < \gamma,\ n < \omega\}$ is a collection of zero-sets of X such that $V_n \cap Z_{i\alpha} \subset W_n \cap H_{i\alpha}$ for each $i \leqslant n, n < \omega$ and $\alpha < \gamma$. We put

$$G^*_\alpha = \bigcup_{n<\omega}\bigcup_{i\leqslant n}(W_n \cap H_{i\alpha}) \quad \text{and} \quad F^*_\alpha = \bigcup_{n<\omega}\bigcup_{i\leqslant n}(V_n \cap Z_{i\alpha})$$

for every $\alpha < \gamma$. Then, $\{G^*_\alpha \mid \alpha < \gamma\}$ is a locally finite collection of cozero-sets of X and $\{F^*_\alpha \mid \alpha < \gamma\}$ is a collection of zero-sets of X such that $F^*_\alpha \subset G^*_\alpha$ for each $\alpha < \gamma$. Moreover, it is easy to see that $F_\alpha \subset F^*_\alpha$ for each $\alpha < \gamma$. Thus, $\{F_\alpha \mid \alpha < \gamma\}$ is uniformly locally finite in X, the proof of (4) is completed. □

It is known that every closed subspace of a space X is P^γ-embedded in X if and only if X is γ-collectionwise normal (e.g., [1]). On the other hand, by [3, Theorem 6.1], every closed subspace of X is z_γ-embedded in X if and only if X is γ-collectionwise normal. Hence, by Propositions 3.1 and 3.5 and these results, we also have that every closed subspace of X is $(P^*)^\gamma$-embedded in X if and only if X is γ-collectionwise normal.

Remark 5.3.

(1) Bing's example G [2] contains a closed C-embedded but not weakly z_{ω_1}-embedded subspace.

(2) Let Y be any countably compact space which is not normal. Then Y contains a closed and non-z-embedded subspace which is weakly z_∞-embedded in Y.

The following proposition is to be compared with results of Hoshina in [9, Section 3] or [8] on unions of P^γ or z_γ-embedded subspaces; note that (1) and (2) do not hold in general for P^γ or z_γ-embedding even in the case of taking finite unions.

Proposition 5.4.

(1) *If A_i is a weakly z_γ-embedded subspace of X for every $i < \omega$, then $\bigcup_{i<\omega} A_i$ is weakly z_γ-embedded in X.*

(2) *If* $\{A_\lambda \mid \lambda \in \Lambda\}$ *is a collection of weakly* z_γ*-embedded subspaces of* X *that has a locally finite expansion of cozero-sets of* X*, then* $\bigcup_{\lambda\in\Lambda} A_\lambda$ *is weakly* z_γ*-embedded in* X.

(3) *Let* $\{A_\lambda \mid \lambda \in \Lambda\}$ *be a uniformly locally finite collection of* $(P^*)^\gamma$*-embedded subspaces of* X*. If* $A_\lambda \cup A_\mu$ *is* C^**-embedded in* X *for every* $\lambda, \mu \in \Lambda$*, then* $\bigcup_{\lambda\in\Lambda} A_\lambda$ *is* $(P^*)^\gamma$*-embedded in* X.

Proof. (1) is easy to show. To prove (2), let $\{A_\lambda \mid \lambda \in \Lambda\}$ be a collection of weakly z_γ-embedded subspaces of X and $\{B_\lambda \mid \lambda \in \Lambda\}$ a locally finite collection of cozero-sets of X such that $A_\lambda \subset B_\lambda$ for each $\lambda \in \Lambda$. Let $\{G_\alpha \mid \alpha < \gamma\}$ be a discrete collection of cozero-sets of $\bigcup_{\lambda\in\Lambda} A_\lambda$ and $\{F_\alpha \mid \alpha < \gamma\}$ a collection of zero-sets of $\bigcup_{\lambda\in\Lambda} A_\lambda$ such that $F_\alpha \subset G_\alpha$ for each $\alpha < \gamma$. Since A_λ is weakly z_γ-embedded in X, there exists a locally finite collection $\mathcal{H}_i^\lambda = \{H_{i\alpha}^\lambda \mid \alpha < \gamma\}$ of cozero-sets of X for each $i < \omega$ such that $F_\alpha \cap A_\lambda \subset \bigcup\{H_{i\alpha}^\lambda \mid i < \omega\}$ for each $\alpha < \gamma$. Let $W_{i\alpha}^\lambda = H_{i\alpha}^\lambda \cap B_\lambda$ for each $\lambda \in \Lambda$, $\alpha < \gamma$ and $i < \omega$. For every $i < \omega$, $\{W_{i\alpha}^\lambda \mid \lambda \in \Lambda, \alpha < \gamma\}$ is a locally finite collection of cozero-sets of X. Hence $\{\bigcup_{\lambda\in\Lambda} W_{i\alpha}^\lambda \mid \alpha < \gamma\}$ is a locally finite collection of cozero-sets of X. Moreover,

$$F_\alpha = \bigcup_{\lambda\in\Lambda} (F_\alpha \cap B_\lambda) \subset \bigcup_{\lambda\in\Lambda} \bigcup_{i<\omega} (H_{i\alpha}^\lambda \cap B_\lambda) = \bigcup_{i<\omega} \left(\bigcup_{\lambda\in\Lambda} W_{i\alpha}^\lambda \right)$$

for each $\alpha < \gamma$. The proof of (2) is completed.

(3) is implied by (2) of Theorem 5.1, (2) of this proposition and [13] (see [9, Theorem 3.18]). $\square$

By Proposition 5.1 and (1) of Proposition 5.4, we have the following result; the case of z_γ-embedding was shown in [8, Theorem 1].

Corollary 5.5. *Let* A_i $(i < \omega)$ *be* z_γ*- (respectively* $(P^*)^\gamma$*-,* P^γ*- or* U^γ*-) embedded subspaces of* X*. Then,* $\bigcup_{i<\omega} A_i$ *is* z_γ*- (respectively* $(P^*)^\gamma$*-,* P^γ*- or* U^γ*-) embedded in* X *if and only if* $\bigcup_{i<\omega} A_i$ *is* z*- (respectively* C^**-,* C*- or* U^ω*-) embedded in* X.

Added in proof.

(1) Recently, Professor V. Gutev kindly sent to the author an e-mail showing that Problem 4.10 is affirmative.

(2) The author recently noticed that, in the definition of weak z_γ-embedding, "uniformly discrete collection $\{F_\alpha \mid \alpha < \gamma\}$ of zero-sets" can be changed into more naturally "uniformly locally finite collection $\{F_\alpha \mid \alpha < \gamma\}$ of subsets" or "locally finite collection $\{F_\alpha \mid \alpha < \gamma\}$ of cozero-sets".

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